Stability of $n-$ Homomorphisms, $n-$ Derivations of A $n-$ Dimensional Additive Functional Equation in C^*- Ternary Algebras

M. Arunkumar[1], T. Namachivayam[2]

Abstract

In this paper, the authors established the generalized Ulam-Hyers stability of $n-$ Homomorphisms, $n-$ Derivations of a $n-$ dimensional additive functional equation

$$\sum_{i=1}^{n} g\left(\sum_{j=1}^{i} x_j\right) = \sum_{i=1}^{n}(n-i+1)\, g(x_i),$$

where $n \geq 2$ on C^*-ternary algebras.

Keywords

Additive functional equation, homomorphism, derivations, C*ternary algebra.

1. Introduction and Preliminaries

The study of stability problems for functional equations is related to a question of Ulam [1] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [2]. It was further generalized and excellent results obtained by number of authors [3,4,5,6,7,8].

During the past two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings, for example, Cauchy-Jensen mappings, k-additive mappings, invariant means, multiplicative mappings, bounded nth differences,convex functions, generalized orthogonality mappings, Euler-Lagrange functional equations, dfferential equations, and Navier-Stokes equations (see[9 -15]).

Also, the stability problem of ternary homomorphisms and ternary derivations was established by Park [16] and J.M.Rassias, Kim [17]

M. Arunkumar, Department of Mathematics, Government Arts College, Tiruvannamalai - 606 603, TamilNadu, India.

T. Namachivayam , Department of Mathematics, Government Arts College, Tiruvannamalai - 606 603, TamilNadu, India.

Definition 1.1 [18] A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x,y,z) \to [x,y,z]$ of A^3 into A, which is $C-$linear in the outer variables, conjugate $C-$linear in the middle variable, and associative in the sense that

$$[x,y,[z,w,v]] = [x,[w,z,y],v] = [[x,y,z],w,v],$$

and satisfies

$$\|[x,y,z]\| - \|x\|\|y\|\|z\| \quad and \quad \|[x,x,x]\| = \|x\|^3$$

Every left Hilbert C^*-module is a C^*-ternary algebra via the ternary product $[x,y,z] = \langle x,y \rangle z$. If a C^*-ternary algebra $(A,[\cdot,\cdot,\cdot])$ has an identity, i.e., an element $e \in A$ such that $x = [x,e,e] = [e,e,x]$ for all $x \in A$, then it is routine to verify that A, endowed with $x \circ y = [x,e,y]$ and $x^* = [e,x,e]$, is a unital C^*-algebra. Conversely, if $(A,\circ)$ is a unital C^*-algebra, then $[x,y,z] = x \circ y^* \circ z$ makes A into a C^*-ternary algebra.

Definition 1.2 [19, 20] Let A and B be C^*-ternary algebras. A $C-$linear mapping $H : A \to B$ is called a C^*-ternary algebra homomorphism if

$$H([x,y,z]) = [H(x),H(y),H(z)]$$

for all $x,y,z \in A$. If, in addition, the mapping H is bijective, then the mapping $H : A \to B$ is called a C^*-ternary algebra isomorphism.

Definition 1.3 [19, 20] A $C-$linear mapping $H : A \to A$ is called a C^*-ternary derivation if

$$\delta([x,y,z]) = [\delta(x),y,z] + [x,\delta(y),z] + [x,y,\delta(z)].$$

Definition 1.4 C^*- ternary algebra $n-$homomorphism Let A and B be C^*-ternary algebras. A $C-$linear mapping $H: A \to B$ is called a C$^*-$ternary algebra homomorphism if

$$H([x_1,\ldots,x_n]) = [H(x_1), H(\ldots), H(x_n)]$$

for all $x_1,\ldots,x_n \in A$.

Definition 1.5 C$^*-$ternary algebra $n-$derivation A $C-$linear mapping $H: A \to A$ is called a C$^*-$ternary derivation if

$$\delta([x_1,\ldots,x_n]) = [\delta(x_1),\ldots,x_n] + \ldots + [x_1,\ldots,\delta(x_n)].$$

In this paper, the authors proved the generalized Ulam-Hyers stability of a $n-$ dimensional additive functional equation

$$\sum_{i=1}^{n} g\left(\sum_{j=1}^{i} x_j\right) = \sum_{i=1}^{n}(n-i+1)g(x_i), \qquad (1.1)$$

where $n \geq 2$ on Banach algebras.

In Section 2 and Section 3, the generalized Ulam - Hyers stability of $n-$homomorphisms and $n-$derivations of a $n-$ dimensional additive functional equation (1.1), is respectively provided.

Through out this paper, let us consider X and Y to be a C^*-ternary algebra with norm $\|\cdot\|_X$ and a C^*-ternary algebra with norm $\|\cdot\|_Y$ respectively.

2. $n-$ Homomorphisms Stability Results

In this section, the generalized Ulam - Hyers stability of $n-$homomorphisms of the additive functional equation (1.1) is provided.

Theorem 2.1 Let $j \in \{-1,1\}$. Assume $\alpha: X^n \to [0,\infty)$ and $\beta: X^n \to [0,\infty)$ be functions such that

$$\lim_{n\to\infty} \frac{\alpha\left(2^{nj}x_1,\ldots,2^{nj}x_n\right)}{2^{nj}} = 0, \qquad (2.1)$$

$$\lim_{n\to\infty} \frac{\beta\left(2^{nj}x_1,\ldots,2^{nj}x_n\right)}{2^{nj}} = 0 \qquad (2.2)$$

for all $x_1,\ldots,x_n \in X$. Let $g: X \to Y$ be a function satisfying the inequality

$$\left\| \sum_{i=1}^{n} g\left(\sum_{j=1}^{i} x_j\right) - \sum_{i=1}^{n}(n-i+1)\, g(x_i) \right\|_Y \leq \alpha(x_1,\ldots,x_n) \qquad (2.3)$$

$$\|g([x_1,\ldots,x_n]) - [g(x_1),\cdots,g(x_n)]\|_Y \leq \beta(x_1,\ldots,x_n) \qquad (2.4)$$

for all $x_1,\ldots,x_n \in X$. Then there exists a unique C^*-ternary algebra n- homomorphism mapping $H: X \to Y$ such that

$$\|g(x) - H(x)\|_Y \leq \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(2^{kj}x, 2^{kj}x, 0\ldots,0)}{2^{kj}} \qquad (2.5)$$

for all $x \in X$.

Proof. Assume $j=1$. Replacing $(x_1, x_2, x_3\ldots,x_n)$ by $(x,x,0\ldots,0)$ in (2.3), we get

$$\left\| g(x) - \frac{g(2x)}{2} \right\|_Y \leq \frac{1}{2(n-1)}\ \alpha(x,x,0,\ldots,0) \qquad (2.6)$$

for all $x \in X$. Now replacing x by $2x$ and dividing by 2 in (2.6), we get

$$\left\| \frac{g(2x)}{2} - \frac{g(2^2x)}{2^2} \right\|_Y \leq \frac{1}{2^2(n-1)} \alpha(2x,2x,0,\ldots,0) \qquad (2.7)$$

for all $x \in X$. From (2.6) and (2.7), we obtain

$$\begin{aligned} &\left\| g(x) - \frac{g(2^2x)}{2^2} \right\|_Y \\ &\leq \left\| g(x) - \frac{g(2x)}{2} \right\|_Y + \left\| \frac{g(2x)}{2} - \frac{g(2^2x)}{2^2} \right\|_Y \\ &\leq \frac{1}{2(n-1)}\left[\alpha(x,x,0,\ldots,0) + \frac{\alpha(2x,2x,0,\ldots,0)}{2} \right] \end{aligned} \qquad (2.8)$$

for all $x \in X$.

In general for any positive integer k, we get

$$\left\| g(x) - \frac{g(2^kx)}{2^k} \right\|_Y$$

$$\leq \frac{1}{2(n-1)}\sum_{i=0}^{k-1}\frac{\alpha(2^i x,2^i x,0,\ldots,0)}{2^i} \qquad (2.9)$$
$$\leq \frac{1}{2(n-1)}\sum_{i=0}^{\infty}\frac{\alpha(2^i x,2^i x,0,\ldots,0)}{2^i}$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{\frac{g(2^k x)}{2^k}\right\}$, replace x by $2^\ell x$ and dividing by 2^ℓ in (2.9), for any $k,\ell > 0$, we deduce

$$\left\|\frac{g(2^\ell x)}{2^\ell}-\frac{g(2^{k+\ell}x)}{2^{(k+\ell)}}\right\|_Y$$
$$=\frac{1}{2^\ell}\left\|g(2^\ell x)-\frac{g(2^k\cdot 2^\ell x)}{2^k}\right\|_Y$$
$$\leq \frac{1}{2(n-1)}\sum_{i=0}^{k-1}\frac{\alpha(2^{i+\ell}x,2^{i+\ell}x,0,\ldots,0)}{2^{i+\ell}}$$
$$\leq \frac{1}{2(n-1)}\sum_{i=0}^{\infty}\frac{\alpha(2^{i+\ell}x,2^{i+\ell}x,0,\ldots,0)}{2^{i+\ell}}$$
$$\to 0 \ as\, k \to \infty$$

for all $x \in X$. Hence the sequence $\left\{\frac{g(2^k x)}{2^k}\right\}$ is Cauchy sequence. Since Y is complete, there exists a mapping $H: X \to Y$ such that

$$H(x)=\lim_{n\to\infty}\frac{g(2^k x)}{2^k},\ \forall\, x\in X.$$

Letting $k\to\infty$ in (2.9) we see that (2.5) holds for all $x \in X$. To prove that H satisfies (1.1), replacing $(x_1,\ldots,x_n)$ by $(2^k x_1,\ldots,2^k x_n)$ and dividing by 2^k in (2.3), we obtain

$$\frac{1}{2^k}\left\|\sum_{i=1}^{n} g\left(\sum_{j=1}^{i}2^k x_j\right)-\sum_{i=1}^{n}(n-i+1)\,g\left(2^k x_i\right)\right\|_Y$$
$$\leq \frac{1}{2^k}\alpha(2^k x_1,\ldots,2^k x_n)$$

for all $x_1,\ldots,x_n \in X$.

Letting $k\to\infty$ in the above inequality and using the definition of $H(x)$, we see that

$$\sum_{i=1}^{n} H\left(\sum_{j=1}^{i}x_j\right)=\sum_{i=1}^{n}(n-i+1)H(x_i).$$

Hence H satisfies (1.1) for all $x_1,\ldots,x_n \in X$. It follows from (2.4) that

$$\|H([x_1,\ldots,x_n])-[H(x_1),\cdots,H(x_n)]\|_Y$$
$$\leq \frac{1}{2^k}\|g([2^k x_1,\ldots,2^k x_n])-[g(2^k x_1),\cdots,g(2^k x_n)]\|_Y$$
$$\leq \frac{1}{2^k}\beta(2^k x_1,\ldots,2^k x_n)$$
$$\to 0 \quad as \qquad k\to\infty$$

for all $x_1,\ldots,x_n \in X$. Hence

$$H([x_1,\ldots,x_n])=[H(x_1),\cdots,H(x_n)]$$

for all $x_1,\ldots,x_n \in X$. To prove that H is unique, let $G(x)$ be another mapping satisfying (2.1) and (2.5), then

$$\|H(x)-G(x)\|_Y$$
$$\leq \frac{1}{2^k}\left\{\|H(2^k x)-g(2^k x)\|_Y+\|g(2^k x)-G(2^k x)\|_Y\right\}$$
$$\leq \frac{1}{2^k}\left\{\frac{1}{2}\sum_{i=0}^{\infty}\frac{\alpha(2^{i+k}x,2^{i+k}x,0,\ldots,0)}{2^{(i+k)}}\right.$$
$$\left.+\frac{1}{2}\sum_{i=0}^{\infty}\frac{\alpha(2^{i+k}x,2^{i+k}x,0,\ldots,0)}{2^{(i+k)}}\right\}$$

$$\leq \sum_{i=0}^{\infty}\frac{\alpha(2^{i+k}x,2^{i+k}x,0,\ldots,0)}{2^{(i+k)}}$$
$$\to 0 \ as\, k\to\infty$$

for all $x \in X$. Hence H is unique. Thus the mapping $H: X \to Y$ is a unique C^*-ternary algebra n - homomorphism satisfying (2.5).

For $j=-1$, we can prove a similar stability result. This completes the proof of the theorem.

The following Corollary is an immediate consequence of Theorem 2.1 concerning the stability of (1.1).

Corollary 2.2 Let λ,λ_1 and s be nonnegative real numbers. Let a function $g: X \to Y$ satisfies the inequality

$$\left\|\sum_{i=1}^{n} g\left(\sum_{j=1}^{i} x_j\right) - \sum_{i=1}^{n}(n-i+1)\, g(x_i)\right\|_Y \leq \begin{cases} \lambda, \\ \lambda \sum_{i=1}^{n} \| x_i \|_X^s, & s<1 \quad or \quad s>1 \\ \lambda\left\{\prod_{i=1}^{n} \| x_i \|_X^s + \sum_{i=1}^{n} \| x_i \|_X^{3s}\right\}, s<\frac{1}{3} & or \quad s>\frac{1}{3}; \end{cases} \tag{2.10}$$

$$\left\|g([x_1,\ldots,x_n]) - [g(x_1),\cdots,g(x_n)]\right\|_Y \leq \begin{cases} \lambda_1, \\ \lambda_1 \sum_{i=1}^{n} \| x_i \|_X^s, \\ \lambda_1\left\{\prod_{i=1}^{n} \| x_i \|_X^s + \sum_{i=1}^{n} \| x_i \|_X^{3s}\right\}, \end{cases} \tag{2.11}$$

for all $x_1,\ldots,x_n \in X$. Then there exists a unique C^*-ternary algebra n- homomorphism function $H: X \to Y$ such that

$$\left\|g(x) - H(x)\right\|_Y \leq \begin{cases} \dfrac{\lambda}{n-1}, \\ \dfrac{\lambda \| x \|_X^s}{(n-1)\,|2-2^s|}, \\ \dfrac{\lambda \| x \|_X^{3s}}{(n-1)\,|2-2^{3s}|}, \end{cases} \tag{2.12}$$

for all $x \in X$.

3. $n-$ Derivations Stability Results

In this section, the generalized Ulam - Hyers stability of $n-$derivations of the additive functional equation (1.1) is given.

Theorem 3.1 Let $j \in \{-1,1\}$. Assume $\alpha: X^n \to [0,\infty)$ and $\beta: X^n \to [0,\infty)$ be functions such that

$$\lim_{n\to\infty} \frac{\alpha\left(2^{nj}x_1,\ldots,2^{nj}x_n\right)}{2^{nj}} = 0, \tag{3.1}$$

$$\lim_{n\to\infty} \frac{\beta\left(2^{nj}x_1,\ldots,2^{nj}x_n\right)}{2^{nj}} = 0 \tag{3.2}$$

for all $x_1,\ldots,x_n \in X$. Let $g: X \to Y$ be a function satisfying the inequality

$$\left\|\sum_{i=1}^{n} g\left(\sum_{j=1}^{i} x_j\right) - \sum_{i=1}^{n}(n-i+1)\, g(x_i)\right\|_Y \leq \alpha(x_1,\ldots,x_n) \tag{3.3}$$

$$\|g([x_1,x_2,\ldots,x_n]) - [g(x_1),x_2,\ldots,x_n] - [x_1,g(x_2),\ldots,x_n] - \cdots - [x_1,x_2,\ldots,g(x_n)]\|_Y \leq \beta(x_1,x_2,\ldots,x_n) \tag{3.4}$$

for all $x_1,\ldots,x_n \in X$. Then there exists a unique C^*-ternary algebra n-derivation mapping $\delta: X \to Y$ such that

$$\left\|g(x) - \delta(x)\right\|_Y \leq \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(2^{kj}x, 2^{kj}x, 0\ldots,0)}{2^{kj}} \tag{3.5}$$

for all $x \in X$.

Proof. It follows from (3.4) that

$$\begin{aligned} &\left\|\delta([x_1,\ldots,x_n]) - [\delta(x_1),\ldots,x_n] - \cdots - [x_1,\ldots,\delta(x_n)]\right\|_Y \\ &\leq \frac{1}{2^k}\left\|g([2^k x_1,\ldots,2^k x_n]) - [\delta(2^k x_1),\ldots,2^k x_n] - \cdots - [2^k x_1,\ldots,\delta(2^k x_n)]\right\|_Y \\ &\leq \frac{1}{2^k}\beta(2^k x_1,\ldots,2^k x_n) \\ &\to 0 \quad as \quad k\to\infty \end{aligned}$$

for all $x_1,\ldots,x_n \in X$. Hence

$$\delta([x_1,\ldots,x_n]) = [\delta(x_1),\ldots,x_n] - \cdots - [x_1,\ldots,\delta(x_n)]$$

for all $x_1,\ldots,x_n \in X$. The rest of the proof is similar tracing to that of Theorem 2.1.

The following Corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.1).

Corollary 3.2 Let λ, λ_1 and s be nonnegative real numbers. Let a function $g : X \to Y$ satisfies the inequality

$$\left\| \sum_{i=1}^{n} g\left(\sum_{j=1}^{i} x_j \right) - \sum_{i=1}^{n} (n-i+1) g(x_i) \right\|_Y \leq \begin{cases} \lambda, \\ \lambda \sum_{i=1}^{n} \| x_i \|_X^s, & s < 1 \quad or \quad s > 1 \\ \lambda \left\{ \prod_{i=1}^{n} \| x_i \|_X^s + \sum_{i=1}^{n} \| x_i \|_X^{3s} \right\}, & s < \frac{1}{3} \quad or \quad s > \frac{1}{3}; \end{cases} \tag{3.6}$$

$$\| g([x_1,\dots,x_n]) - [g(x_1),\dots,x_n] - \cdots - [x_1,\dots,g(x_n)] \|_Y \leq \begin{cases} \lambda_1, \\ \lambda_1 \sum_{i=1}^{n} \| x_i \|_X^s, \\ \lambda_1 \left\{ \prod_{i=1}^{n} \| x_i \|_X^s + \sum_{i=1}^{n} \| x_i \|_X^{3s} \right\}, \end{cases} \tag{3.7}$$

for all $x_1,\dots,x_n \in X$. Then there exists a unique $C^* -$ternary algebra n- derivation function $\delta : X \to Y$ such that

$$\| g(x) - \delta(x) \|_Y \leq \begin{cases} \frac{\lambda}{n-1}, \\ \frac{\lambda \| x \|_X^s}{(n-1)|2-2^s|}, \\ \frac{\lambda \| x \|_X^{3s}}{(n-1)|2-2^{3s}|}, \end{cases} \tag{3.8}$$

for all $x \in X$.

4. Conclusion

The additive function $g(x) = x$ is the solution of the additive functional equation (1.1), the functional equation can be rewritten as follows

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{i} x_j \right) = \sum_{i=1}^{n} (n-i+1)(x_i).$$

That is

$$x_1 + (x_1 + x_2) + \cdots + (x_1 + x_2 + \cdots + x_n) = n x_1 + (n-1) x_2 + \cdots\cdots + x_n.$$

If we replace, the "$+$" by "$\vee$" in the above identity, then the truth values satisfies the equivalence relation.

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Dr. M. Arunkuamr is currently working as an assistant professor in the Department of Mathematics, Government Arts College, Tiruvannamalai. He is a life member in Indian Mathematical Society and he has published 75 papers in National and international journals.

Prof. T. Namachivayam is currently working as an assistant professor in the Department of Mathematics, Government Arts College, Tiruvannamalai.